General Equilibrium under Uncertainty

Uncertainty and General Equilibrium

- In Chapter 7 of the *Theory of Value*, Debreu extends the analysis of general equilibrium (optimality, existence) to allow for uncertainty.
- He assumes two dates: the current date $t = 0$ and the future date $t = 1$.
- At $t = 1$, different states of nature can happen, say two states α and β.
- The subjective probabilities of the two states are given by $\pi_i(\alpha)$ and $\pi_i(\beta)$.
- Every agent has a von Newmann & Morgenstern expected utility defined over his consumption at date 0, $c_i(0)$, at date 1 state α, $c_i(\alpha)$ and at date 1 state β, $c_i(\beta)$,

$$\begin{aligned} U_i(c_i(0), c_i(\alpha), c_i(\beta)) &= \pi_i(\alpha)u_i(c_i(0), c_i(\alpha)) \\ &+ \pi_i(\beta)u_i(c_i(0), c_i(\beta)). \end{aligned}$$

Contingent Commodities

- We assume that at date $t = 0$, markets open for the exchange of all contingent commodities $x_i(\alpha)$ and $x_i(\beta)$.
- A *contingent commodity* is a commodity, which will be consumed in state s if the state s realizes.
- The idea is that you can, at date 0, trade commodities which will become available at different states at date 1.

General Equilibrium with Contingent Commodities

- A general equilibrium with contingent commodities is an extension of the general equilibrium model with new goods.

- Formally, an equilibrium is a vector of prices p^* and an allocation x^* such that:

  1. every agent i maximizes $\pi_i(\alpha)u_i(c_i(0), c_i(\alpha)) + \pi_i(\beta)u_i(c_i(0), c_i(\beta))$ subject to $p(0)(x_i(0)-e_i(0))+p(\alpha)(x_i(\alpha)-e_i(\alpha))+p(\beta)(x_i(\beta)-e_i(\beta)) = 0$;
  2. (markets clear) $\sum_i x_i(0) = \sum_i e_i(0)$, $\sum_i x_i(\alpha) = \sum_i e_i(\alpha)$, $\sum_i x_i(\beta) = \sum_i e_i(\beta)$;
  3. an equilibrium always exists and is optimal.

General Equilibrium with Contingent Commodities (Remarks)

- There are markets for every contingent commodity.
- Deliveries are contingent, prices are not!
- We assume strict convexity in preferences.
- All individuals must have the same information, and all know that all have the same information, and all know that all know .... (i.e. common knowledge).
- Although the model remains static, there are implicit dynamics as decisions are taken before uncertainty is resolved (*ex ante*), but trade is implemented after the state of nature is known (*ex post*).

Example

- Suppose that there are two states and that the consumer does not consume in state 0.
- Utilities are then given by
$$U_i = \pi_i(\alpha)u_i(c_i(\alpha)) + \pi_i(\beta)u_i(c_i(\beta)).$$
- Consumers have endowments $e_i(\alpha), e_i(\beta)$.

Example (Cont.)

- Suppose that there are two consumers, 1 and 2.
- In addition, *total endowments are the same in the two states* (no aggregate risk); that is,

$$e_1(\alpha) + e_2(\alpha) = e_1(\beta) + e_2(\beta).$$

- Recall the maximization problem for, say agent 1.

$$U_1 = \pi_1(\alpha)u_1(c_1(\alpha)) + \pi_1(\beta)u_1(c_1(\beta))$$

subject to

$$p(\alpha)(x_1(\alpha) - e_1(\alpha)) + p(\beta)(x_1(\beta) - e_1(\beta)) = 0.$$

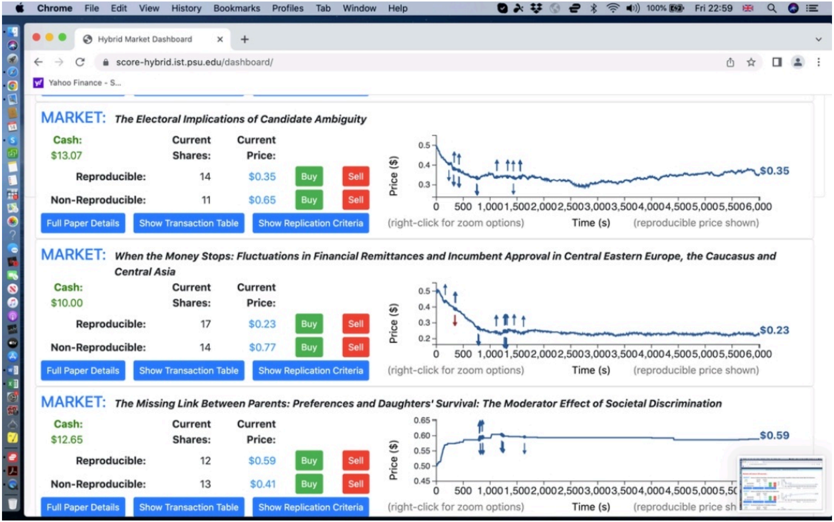
- The first order condition gives:

$$\frac{\pi_1(\alpha)u_1'(c_1(\alpha))}{\pi_1(\beta)u_1'(c_1(\beta))} = \frac{p(\alpha)}{p(\beta)} = \frac{\pi_2(\alpha)u_2'(c_2(\alpha))}{\pi_2(\beta)u_2'(c_2(\beta))}.$$

Arrow Securities

- If there are many goods, the contingent commodities model implies a huge number of spot markets (trading good l in state α with good k in state β).
- There is an alternative, equivalent but much simpler way to obtain the same result.
- An *Arrow security* is a financial asset which pays 1 unit of numéraire in some state and 0 in all other states.
- Arrow securities are now traded on the market with a price q in period 0.

Prediction Markets



Complete Arrow Securities

- Suppose that there are as many Arrow securities as there are states.
- With two states α and β, this means two Arrow securities; one paying 1 unit in state α and 0 in state β, while the other paying 1 unit in state β and 0 in state α.
- Agents choose b^1 and b^2, the quantity of the two Arrow securities that they buy for q^1 and q^2, respectively, in period 0.
- In period 1, agents collect the returns of the Arrow securities (i.e. b^1 and b^2) and buy consumption goods $x(\alpha)$ (or $x(\beta)$).

Budget Constraints

- Consumers face three budget constraints:

$$\begin{aligned} x_i(0) - e_i(0) &= -q^1 b_i^1 - q^2 b_i^2, \\ p(\alpha)(x_i(\alpha) - e_i(\alpha)) &= b_i^1, \\ p(\beta)(x_i(\beta) - e_i(\beta)) &= b_i^2. \end{aligned}$$

Equivalence Result

- We can prove *equivalence* between a market with complete Arrow securities and a market with contingent commodities.
- Replacing b_i^1 and b_i^2 from the second and third budget constraints into the first one, we get a unique budget constraint

  $$(x_i(0)-e_i(0))+q^1p(\alpha)(x_i(\alpha)-e_i(\alpha))+q^2p(\beta)(x_i(\beta)-e_i(\beta))$$

  as in the case of contingent commodities (see slide 4).
- We also conclude that an equilibrium exists and is optimal.

Rational Expectations

- Assume now that different consumers have different information partitions.
- The price p aggregates information detained by different consumers.
- In a *Rational Expectations equilibrium*, all consumers understand the mapping from state s to the price p_s.
- They can invert that mapping to find the state.
- An equilibrium is called *revealing* if all consumers can learn the state through prices.

Example

- Suppose that there are two states, α and β with objective probabilities $\frac{1}{2}, \frac{1}{2}$, respectively.
- There are two consumers, $i = 1, 2$.
- There are two goods with prices $p(s)$ and $q(s)$.
- The prices of the goods are normalized so that $p(s) + q(s) = 1$, $s = \alpha, \beta$.

Example (Cont.)

- The first agent knows the state of nature. His endowments are $e_1(\alpha) = (1, 0), e_1(\beta) = (0, 1)$.
- His utility is $U_1(x_1(s), y_1(s)) = \sqrt{x_1(s)y_1(s)}$, $s = \alpha, \beta$.
- The second agent does not know the state of nature and has endowments $e_2(\alpha) = e_2(\beta) = (1, 1)$.
- His utility is $U_2(x_2, y_2) = \frac{1}{4}\log x_2 + \frac{3}{4}\log y_2$ in state α and $U_2(x_2, y_2) = \frac{3}{4}\log x_2 + \frac{1}{4}\log y_2$ in state β.
- Crucially, observe that given that consumer 2 does not know the state of nature, this implies that the equilibrium here is not revealing (i.e. $p(\alpha) = p(\beta)$).

How to Find the Competitive Equilibrium?

It is straight-forward to find the competitive equilibrium.

1. Derive demand for fixed prices.
2. Use the market clearing conditions to compute equilibrium prices.

Recall the market clearing conditions.

$x_1(\alpha) + x_2(\alpha) = e_1^x(\alpha) + e_2^x(\alpha)$;

$y_1(\alpha) + y_2(\alpha) = e_1^y(\alpha) + e_2^y(\alpha)$;

$x_1(\beta) + x_2(\beta) = e_1^x(\beta) + e_2^x(\beta)$;

$y_1(\beta) + y_2(\beta) = e_1^y(\beta) + e_2^y(\beta)$.

Example (Cont.)

- The first consumer's demand can be computed as

$$x_1(\alpha) = \frac{1}{2}, \qquad y_1(\alpha) = \frac{p(\alpha)}{2q(\alpha)}, \quad \text{and}$$

$$x_1(\beta) = \frac{q(\beta)}{2p(\beta)}, \qquad y_1(\beta) = \frac{1}{2}.$$

Given that the agent is informed, the objective function of the optimization problem is formulated assuming a specific state for the agent.

$x_1(\alpha)$ & $y_1(\alpha)$

Max $\sqrt{x_1(s)y_1(s)}$ subject to

$p(s)x_1(s) + q(s)y_1(s) = p(s)e_1^x(s) + q(s)e_1^y(s)$ for $s = \alpha, \beta$.

Assume $s = \alpha$**.**

Thus, $p(\alpha)x_1(\alpha) + q(\alpha)y_1(\alpha) = p(\alpha)$ as $e_1(\alpha) = (1, 0)$.

[1]: $\frac{1}{2}\frac{y_1(a)^{\frac{1}{2}}}{x_1(a)^{\frac{1}{2}}} - \lambda p(a) = 0$;

[2]: $\frac{1}{2}\frac{x_1(a)^{\frac{1}{2}}}{y_1(a)^{\frac{1}{2}}} - \lambda q(a) = 0$;

[3]: $p(a)[x_1(a) - 1] + q(a)y_1(a) = 0$.

Dividing [1] by [2]: $\frac{y_1(a)}{x_1(a)} = \frac{p(a)}{q(a)} \to x_1(a) = \frac{y_1(a)q(a)}{p(a)}$.

Substituting the last expression in [3] gives us that $y_1(\alpha) = \frac{p(\alpha)}{2q(\alpha)}$ and then that $x_1(\alpha) = \frac{1}{2}$.

Example (Cont.)

- If the second agent remains uninformed, his demand is given by

$$x_2(s) = \frac{1}{2p(s)}, \quad y_2(s) = \frac{1}{2q(s)}$$

for $s = \alpha, \beta$.

Given that the agent is uninformed, the objective function of the optimization problem is formulated assuming the expected utility of the agent.

x_2 & y_2

Max $\frac{1}{2}[\frac{1}{4}log\ x_2 + \frac{3}{4}log\ y_2] + \frac{1}{2}[\frac{3}{4}log\ x_2 + \frac{1}{4}log\ y_2]$
$= \frac{1}{2}log\ x_2 + \frac{1}{2}log\ y_2$

subject to

$px_2 + qy_2 = pe_2^x + qe_2^y$.

Thus, $px_2 + qy_2 = p + q$ as $e_2 = (1, 1)$.

[1]: $\frac{1}{2}\frac{1}{x_2} - \lambda p = 0$;

[2]: $\frac{1}{2}\frac{1}{y_2} - \lambda q = 0$;

[3]: $p[x_2 - 1] + q[y_2 - 1] = 0$.

Dividing [1] by [2]: $\frac{y_2}{x_2} = \frac{p}{q} \to x_2 = \frac{qy_2}{p}$.

Substituting the last expression in [3] gives us that $y_2 = \frac{p+q}{2q} = \frac{1}{2q}$ and then that $x_2 = \frac{1}{2p}$.

Example (Cont.)

- The equilibrium prices are calculated using the market clearing conditions:

$$p(\alpha) = \frac{1}{3}, \quad q(\alpha) = \frac{2}{3}, \quad p(\beta) = \frac{2}{3}, \quad q(\beta) = \frac{1}{3}.$$

- **In sharp contrast to what was assumed, we have that $p(\alpha) \neq p(\beta)$; hence, this is a revealing equilibrium.**

$p(\alpha)$ & $q(\alpha)$

Recall that $x_1(\alpha) + x_2(\alpha) = e_1^x(\alpha) + e_2^x(\alpha)$.

Thus, $\frac{1}{2} + \frac{1}{2p(\alpha)} = 2 \rightarrow \frac{3}{2} = \frac{1}{2p(\alpha)} \rightarrow p(\alpha) = \frac{1}{3}$.

Recall that $y_1(\alpha) + y_2(\alpha) = e_1^y(\alpha) + e_2^y(\alpha)$.

Thus, $\frac{p(\alpha)}{2q(\alpha)} + \frac{1}{2q(\alpha)} = 1 \rightarrow \frac{4}{3} = 2q(\alpha) \rightarrow q(\alpha) = \frac{2}{3}$.

Example (Cont.)

- Let's assume from the beginning that this is a revealing equilibrium; thus, consumer 2 learns the state.

- Therefore, consumer 2 formulates the problem like consumer 1 and, as a result, has demands

$$x_2(\alpha) = \frac{1}{4p(\alpha)}, \qquad y_2(\alpha) = \frac{3}{4q(\alpha)}, \quad \text{and}$$
$$x_2(\beta) = \frac{3}{4p(\beta)}, \qquad y_2(\beta) = \frac{1}{4q(\beta)}.$$

- The equilibrium prices are

$$p(\alpha) = \frac{1}{6}, \;\; q(\alpha) = \frac{5}{6}, \;\; p(\beta) = \frac{5}{6}, \;\; q(\beta) = \frac{1}{6}.$$

Inexistence of Rational Expectations Equilibrium

- There are examples where the Rational Expectations equilibirum does not exist.
- This happens when, assuming that the equilibrium is not revealing, we obtain different prices in the two states.
- But assuming that the equilibrium is revealing, we obtain a single price in the two states.

Example

- Assume that

$$\begin{aligned} U_1(x,y,\alpha) &= \frac{2}{3}\log x + \frac{1}{3}\log y, \\ U_1(x,y,\beta) &= \frac{1}{3}\log x + \frac{2}{3}\log y, \\ U_2(x,y,\alpha) &= \frac{1}{3}\log x + \frac{2}{3}\log y, \\ U_2(x,y,\beta) &= \frac{2}{3}\log x + \frac{1}{3}\log y. \end{aligned}$$

- Agent 1 knows the state but not agent 2.
- Endowments are $e_1 = e_2 = (1,1)$ in both states.

Example (Cont.)

- Suppose that the equilibrium is not revealing (i.e. $p(\alpha) = p(\beta)$). We have

$$x_1(\alpha) = \frac{2}{3p(\alpha)}, \qquad y_1(\alpha) = \frac{1}{3q(\alpha)},$$
$$x_1(\beta) = \frac{1}{3p(\beta)}, \qquad y_1(\beta) = \frac{2}{3q(\beta)},$$
$$x_2(s) = \frac{1}{2p(s)}, \qquad y_2(s) = \frac{1}{2q(s)},$$

- which result in equilibrium prices

$$p(\alpha) = \frac{7}{12}, \quad q(\alpha) = \frac{5}{12}, \quad p(\beta) = \frac{5}{12}, \quad q(\beta) = \frac{7}{12}.$$

- We have that $p(\alpha) \neq p(\beta)$ but we are told that this is not a revealing equilibrium. Therefore, we need to reformulate the maximization problem for agent 2.

Example (Cont.)

- Suppose that the equilibrium is revealing (i.e. $p(\alpha) \neq p(\beta)$). We have

$$\begin{aligned} x_1(\alpha) &= \frac{2}{3p(\alpha)}, \qquad & y_1(\alpha) &= \frac{1}{3q(\alpha)}, \\ x_1(\beta) &= \frac{1}{3p(\beta)}, \qquad & y_1(\beta) &= \frac{2}{3q(\beta)}, \\ x_2(\alpha) &= \frac{1}{3p(\alpha)}, \qquad & y_2(\alpha) &= \frac{2}{3q(\alpha)}, \\ x_2(\beta) &= \frac{2}{3p(\beta)}, \qquad & y_2(\beta) &= \frac{1}{3q(\beta)}, \end{aligned}$$

- which result in equilibrium prices

$$p(\alpha) = \frac{1}{2}, \quad q(\alpha) = \frac{1}{2}, \quad p(\beta) = \frac{1}{2}, \quad q(\beta) = \frac{1}{2}.$$

- We have that $p(\alpha) = p(\beta)$; hence, this is not a Rational Expectations equilibrium.

Summary

- The general equilibrium model can be extended to take into account contingent commodities.
- The model with contingent commodities is equivalent to a simpler model where Arrow securities are traded.
- If agents have different information, prices can convey information and the equilibrium concept is a Rational Expectations equilibrium.
- The Rational Expectations equilibrium may fail to exist.